The Chain Rule for Composite Functions

If $y = f(g(x))$, then the chain rule must be used to find its derivatives.

If f and g are differentiable functions, then

$$\frac{d}{dx}[f(g(x)] = f'(g(x)) \cdot g'(x)$$

That is, **the derivative of a composition of functions is the product of the derivative of the "outside function", f, and the derivative of the "inside function", g.**

The chain rule can be simplified using a substitution for the "inside function".

Let $u = g(x)$, then $f(g(x)) = f(u)$ and

$$\frac{d}{dx}[f(g(x))] = \frac{d}{du}[f(u)] = \frac{df}{du} \cdot \frac{du}{dx}$$

Note that when u is substituted for the *inside function* f becomes a function of u so the derivative of f is $\frac{df}{du}$ and since $g(x) = u$ (u is a function of x), $g' = u' = \frac{du}{dx}$.

Example 1

Find the derivative of $y = \left(x^2 + 1\right)^5$.

Since $y = \left(x^2 + 1\right)^5$ is a composite function we must use the Chain Rule.

Let $g(x) = x^2 + 1$ be the *inside function* and then $f(x) = x^5$ would be the *outside function*

Let $u = x^2 + 1$ for the *inside function* then $f(x) = (x^2 + x)^5$ becomes $f(u) = u^5$. Now use the chain rule $\frac{d}{du}[f(u)] = \frac{df}{du} \cdot \frac{du}{dx}$ to find the derivative of $y = (x^2 + x)^5$.

$$\frac{d}{dx}\left[(x^2 + x)^5\right] = \frac{d}{du}\left[u^5\right] \cdot \frac{d}{dx}[u]$$
$$= 5u^4 \cdot (2x + 1)$$
$$= 5(x^2 + x)^4(2x + 1)$$

Example 2: Find y' for $y = e^{x^2 + x}$. Let $u = x^2 + x$, then

$$\frac{d}{dx}\left[e^{x^2+x}\right] = \frac{d}{dx}\left[e^u\right]$$
$$= \frac{d}{du}\left[e^u\right] \cdot \frac{d}{dx}\left[x^2 + x\right]$$
$$= e^u(2x + 1)$$
$$= e^{x^2+x}(2x + 1)$$

Example 3: Find f' for $f(x) = 5^{x^2+x}$ without substitution.

The *inside function* for $f(x) = 5^{x^2+x}$ **is** $x^2 + x$**.** Recall that $\frac{d}{dx}\left[5^x\right] = (\ln 5)\cdot 5^x$. To find f' without substitution, think "the derivative of the exponential function base 5" times "the derivative of the exponent of the base 5, $x^2 + x$**".** That is,

$$\begin{aligned}\frac{d}{dx}\left[5^{x^2+x}\right] &= \frac{d}{dx}\left[5^{x^2+x}\right]\cdot\frac{d}{dx}\left[x^2+x\right]\\ &= (\ln 5)5^{x^2+x}(2x+1)\\ &= (2x+1)(\ln 5)5^{x^2+x}\end{aligned}$$

Example 4: Exercise 12, Page 126. Find g' for $g(x) = 3^{(2x+7)}$.

$$\begin{aligned}\frac{d}{dx}\left[3^{2x+7}\right] &= \frac{d}{dx}\left[3^{2x+7}\right]\cdot\frac{d}{dx}\left[2x+7\right]\\ &= (\ln 3)\cdot 3^{2x+7} + (2)\\ &= 2 + (\ln 3)\cdot 3^{2x+7}\end{aligned}$$

Example 5: Exercise 23, Page 126. Find $\frac{dy}{ds}$ for $y = \sqrt{s^3+1}$. Note that $\sqrt{s^3+1} = \left(s^3+1\right)^{1/2}$.

$$\begin{aligned}\frac{dy}{ds} &= \frac{d}{ds}\left[\sqrt{s^3+1}\right]\\ &= \frac{d}{ds}\left[\left(s^3+1\right)^{1/2}\right]\cdot\frac{d}{ds}\left[s^3+1\right]\\ &= \frac{1}{2}\left(s^3+1\right)^{1/2-1}\cdot(3s^2)\\ &= \frac{1}{2}\left(s^3+1\right)^{-1/2}\cdot(3s^2)\\ &= \frac{3s^2}{2\sqrt{s^3+1}}\end{aligned}$$

Example 6: Exercise 46, Page 126. Find f' for $f(x) = (ax^2+b)^3$.

$$\begin{aligned}f'(x) &= \frac{d}{dx}\left[(ax^2+b)^3\right]\cdot\frac{d}{dx}\left[(ax^2+b)\right]\\ &= 3(ax^2+b)^2\cdot(2ax)\\ &= 6ax(ax^2+b)^2\end{aligned}$$